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The IR/UV connection in non-commutative gauge theories

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Abstract: Quantum field theory on non-commutative spaces does not enjoy the usual ultraviolet-infrared decoupling that forms the basis for conventional renormalization. The high momentum contributions to loop integrations can lead to unfamiliar long distance behavior which can potentially undermine naive expectations for the IR behavior of the theory. These "anomalies" involve non-analytic behavior in the noncommutativity parameter θ making the limit $\theta \to 0$ singular. In this paper we will analyze such effects in the one loop approximation to gauge theories on non-commutative space. We will see that contrary to expectations poles in θ do occur and lead to large discrepancies between the expected and actual infrared behavior. We find that poles in θ are absent in supersymmetric theories. The "anomalies" are generally still present, but only at the logarithmic level. A notable exception is non-commutative super Yang-Mills theory with 16 real supercharges in which anomalous effects seem to be absent altogether.

Keywords: Gauge Symmetry, Non-Commutative Geometry.

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Contents

1. The UV/IR connection in non-commutative field theory

Field theories formulated on non-commutative spaces are interesting in both their own right as well as for their applications to string and matrix theories [1, 2]. These theories are characterized by a non-commutativity parameter θ with dimensions of length squared. Classically and in the tree-level approximation the behavior of the theory for momenta much less than $\theta^{-1/2}$ is the same as for the corresponding commutative theory. However this is not necessarily the case for the quantum theory. The non-commutativity can lead to unfamiliar effects of the ultraviolet modes on the infrared behavior which have no analog in conventional quantum field theory [3].

The origin of the strange mixing of IR and UV effects in non-commutative field theory can be understood in a simple way [4, 5]. The field quanta in such a theory can be thought of as pairs of opposite charges moving in a strong magnetic field. The spatial locations of the two charges are defined by a center of mass position x^i_{cm} and a relative coordinate Δ^m. The relative coordinate is related to the spatial momentum p by

$$\Delta^i = \theta^{ij} p_j \,, \tag{1.1}$$

where θ is an antisymmetric matrix with components in the spatial directions. In this paper we will consider the case of 3-dimensional space. Without loss of generality θ can be taken to lie in the $(1,2)$ plane

$$\begin{aligned} \theta^{1,2} &= -\theta^{2,1} \equiv \theta \,, \\ \theta^{1,3} &= \theta^{3,2} = 0 \,. \end{aligned} \tag{1.2}$$

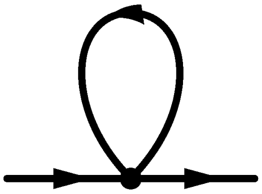

Figure 1: Mass renormalization correction in ϕ^4 thoery.

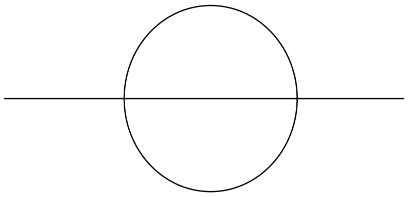

Figure 2: Non-planar correction in non-commutative ϕ^4 thoery.

The momentum in the $(1,2)$ plane will be called P. We will also use the notation $\tilde{P}_i = \theta_{ij} P^j$. Thus a particle moving with momentum P along the X^1 axis has a spatial extension of size $|\theta P|$ in the X^2 direction. The growth of the size of a particle with its momentum has interesting consequences. For example, when a quantum of momentum P scatters off a target at rest, the scattering amplitude will spread in impact parameter space over a distance $|\theta P|$.

There are also important and somewhat bizarre consequences for Feynman loop integrations. Roughly speaking, when a particle of momentum P circulates in a loop it can induce an effect at distance $|\theta P|$. The high momentum end of the integrals can give rise to power law long range forces which are absent entirely in the classical theory. We will call such effects "anomalies" although we emphasize that they do not signal an inconsistency in the theory but rather a violation of naive expectations.

In the case of planar diagrams the effect of non-commutativity is simple. Every diagram gets multiplied by a phase factor that depends only on the momenta of the external lines. Thus the Feynman integrals are exactly as in the commutative theory. In the non-planar case (for example, figure 2) the situation is more interesting. The phase factors now involve products of external momenta p and internal momenta l in the form

$$e^{ip\theta l} = e^{ip\tilde{l}} \,. \tag{1.3}$$

If the diagram in question is divergent in the commutative theory, the effect of the oscillating phases is typically to regulate the diagram and render it finite. But as $P \to 0$ the phases become ineffective and the diagram diverges at $P = 0$. This is the mechanism described in detail in [3]. We will begin by reviewing a simple example from non-commutative ϕ^4 theory.

The diagrams in question, figures 1 and 2, are the lowest order mass renormalization corrections to the propagator. We are interested in the non-planar contribution which in the commutative theory has the form

$$\int d^4 l \, \frac{1}{l^2} \,. \tag{1.4}$$

The diagram is quadratically divergent and is renormalized by a mass counter term. In the commutative theory, the Feynman integrals involved in the planar and non-planar graphs are the same.

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In the non-commutative case the integrand for the diagram in figure 2 has an additional factor $\exp(i\tilde{p}l)$ where p and l are the external[1] and loop momenta. The integral has the form

$$\int d^4l \, \frac{1}{l^2} \, e^{i\tilde{p}l} \sim \frac{1}{\theta^2 P^2} \, . \tag{1.5}$$

As emphasized in [3] there are some very striking features of this result. The first is that the pole at $P = 0$ arises from the high momentum region of integration. Although we evaluated it for the massless theory, the pole itself is independent of mass. Furthermore this contribution to the self energy has a huge effect on the propagation of long wavelength particles. The on-shell condition or dispersion relation becomes

$$p_0^2 = p_3^2 + P^2 + c\frac{1}{\theta^2 P^2} \, , \tag{1.6}$$

where c is proportional to the coupling constant. Thus, as discussed in [3], the behavior of the non-commutative theory below the non-commutativity scale seems to be nothing like the commutative theory. In this case the low momentum end of the spectrum is completely removed from the low energy theory.

Commutative gauge theories are better behaved in the UV than commutative scalar theories. The worst divergences in pure Yang-Mills theory or Yang-Mills theory with fermions are logarithmic. This naively suggests that in their non-commutative versions the worst anomalous effects will be logarithmic in P. As an example consider the vacuum polarization correction to the gauge boson propagator. The divergences have the form

$$\Pi \sim g^2 p^2 \log \kappa \, , \tag{1.7}$$

where p^2 is the squared four-momentum of the gauge boson. Note in particular that the mass correction vanishes since Π vanishes at $p^2 = 0$. This situation suggests that in the non-commutative theory the worst anomalous effect in the propagator has the form

$$\Pi \sim g^2 p^2 \log \tilde{P}^2 \, . \tag{1.8}$$

If this were so, the dispersion relation of a low energy gauge boson would be unaffected by the non-commutativity. As we will see in the next section this is generally incorrect.

2. U(1) non-commutative Yang-Mills

In this section we analyze U(1) Yang-Mills theory on a non-commutative space. The classical action is given by

$$S = -\frac{1}{4} \int d^4x F^2 \, , \tag{2.1}$$

[1] In the rest of the paper, we denote by small p the 4-momentum vector with $p_{1,2} \equiv P_{1,2}$ in the non-commutativity plane.

with the field strength F given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{2.2}$$

and

$$[A_\mu, A_\nu] = A_\mu * A_\nu - A_\nu * A_\mu \, . \tag{2.3}$$

The $*$-product between two functions $\phi_1(x)$ and $\phi_2(x)$ is defined by

$$\phi_1 * \phi_2(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial^y_\mu \partial^z_\nu} \phi_1(y)\phi_2(z)|_{y=z=x} \, . \tag{2.4}$$

The theory is invariant under non-commutative gauge transformations

$$\delta_\lambda A_\mu = \partial_\mu \lambda - gi(A_\mu * \lambda - \lambda * A_\mu) \, . \tag{2.5}$$

We may add matter fields in the theory as well. The scalar and fermionic parts of the action that involve interactions of the matter fields with the gauge field are given by

$$S_{\text{matter}} = \int d^4 x i \bar{\psi} * \gamma^\mu D_\mu \psi + \frac{1}{2}(D_\mu \Phi)^2 \, . \tag{2.6}$$

The covariant derivative acts on the fields as follows:

$$D_\mu X = \partial_\mu X - ig[A_\mu, X] \, . \tag{2.7}$$

The commutator is defined through the $*$-product as before. The matter fields are covariant under the family of gauge transformations given in eq. (2.5).

The Feynman rules for the theory have been worked out in refs. [6, 8]. The vertices look similar to those of a commutative non-abelian gauge theory with all matter fields in the adjoint of the gauge group. The structure constants are replaced by sines of external momenta as shown in detail in the appendix A. The Feynman rules for the ghosts are also included. The vertices vanish when θ is taken to be zero as expected. Our computations are done in the Feynman gauge. In this gauge the gauge field propagator is given by

$$-i\frac{g_{\mu\nu}}{p^2} \, . \tag{2.8}$$

We refer the reader in [6, 7] for a derivation of the perturbative Feynman rules and gauge fixing of the theory.

In this paper we are concerned with one-loop parturbative computations only. In particular, we do not study the strongly coupled theory. Our results are strictly properties of the weakly coupled theory.

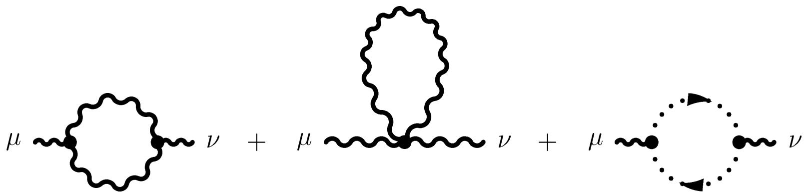

Figure 3: Photon self energy: gauge loop corrections.

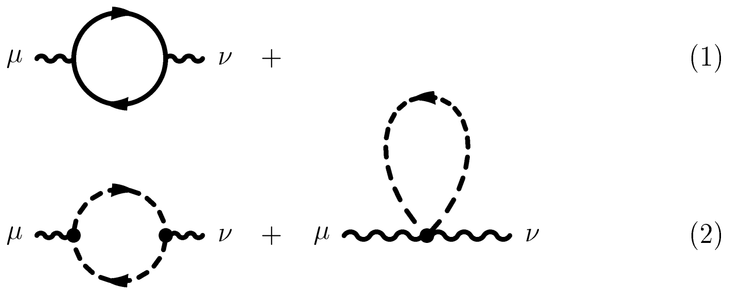

Figure 4: Photon self energy: matter loop corrections.

3. Photon self-energy correction

In non-commutative gauge theory the most serious anomalous effects, namely those which exhibit inverse powers of $\tilde{P}$, are associated with the 2- and 3-point functions. We will begin with the computation of the 2-point photon self energy diagrams. We will consider the contributions from loops involving fermions, scalars and gauge bosons (including ghosts). It should be noted that the individual contributions are gauge invariant. Similar calculations have been done by Hayakawa [8]. The relevant diagrams are shown in figures 3 and 4. We will illustrate the procedure with the fermion loop (line (1) in figure 4).

Using the Feynman rules from appendix A we find

$$i\Pi_f^{\mu\nu}(p) = -4g^2 N_f \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu (\not{l} - \not{p})\gamma^\nu \not{l}\right]}{(l-p)^2 l^2} \sin^2\left(\frac{1}{2}\tilde{p}l\right), \tag{3.1}$$

where N_f is the number of Majorana fermions, each of which counts as two fermion species.

We are interested in the contribution to the integral coming from very high loop momentum. We therefore drop the sub-leading dependence in the integrand and replace eq. (3.1) by

$$i\Pi_f^{\mu\nu}(p) = -4g^2 N_f \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu \not{l} \gamma^\nu \not{l}\right]}{(l)^4} \sin^2\left(\frac{1}{2}\tilde{p}l\right), \tag{3.2}$$

Using

$$\sin^2\left(\frac{1}{2}\tilde{p}l\right) = \frac{1}{2}[1 - \cos(\tilde{p}l)], \tag{3.3}$$

we can isolate the planar and non-planar contributions. The non-planar contribution is obtained by dropping the first term and keeping only the cos term.[2]

Working out the trace, we get

$$i\Pi_f^{\mu\nu}(p) = 4g^2 N_f \int \frac{d^4l}{(2\pi)^4} \frac{(2l^\mu l^\nu - g^{\mu\nu} l^2)}{l^4}\, e^{i\tilde{p}\, l}\,. \tag{3.4}$$

Since $\int d^4l/(2\pi)^4 \frac{1}{l^4}\, e^{i\tilde{p}\, l} = i\alpha\, (\log L - \log|\tilde{p}|)$,[3] where L denotes a long distance cut-off and α is a positive real constant, we can rewrite (3.4) as

$$i\Pi_f^{\mu\nu}(p) = 4ig^2 N_f \alpha \left(2\,\partial^\mu \partial^\nu - g^{\mu\nu}\partial^2\right) \log|\tilde{p}|\,. \tag{3.5}$$

The integral (3.4) is finite, and the cut-off dependence vanishes after differentiating. We note that the term $g^{\mu\nu}/\tilde{p}^2$ cancels in this expression leaving us with

$$i\Pi_f^{\mu\nu}(p) = -8ig^2(2N_f)\alpha\, \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^4}\,. \tag{3.6}$$

The answer is somewhat surprising. If the factor $e^{i\tilde{p}l}$ were not present in eq. (3.4) the expression would be a conventional self energy diagram of the commutative theory.[4] Gauge invariance would be invoked to say that any quadratic divergence is absent. Alternatively the diagram can be Pauli Villars regulated eliminating the quadratic divergence. The integral with the oscillating phase is finite and well defined. However it quadratically diverges as $\tilde{p} \to 0$. Thus we see that there is an anomalous effect of order θ^{-2} arising out of a diagram which in the commutative theory is quadratically divergent by power counting but for which the divergence vanishes as a consequence of symmetry. As noted in [3] in the context of scalar theories this type of behavior is proportional to inverse powers of θ. Evidently the limit in which $\theta \to 0$ does not smoothly tend to the commutative theory.

Here we would like to add some comments. When θ is taken to be zero, the integrand in eq. (3.1) vanishes. This is because the corresponding U(1) commutative theory is a free theory. However, this does not imply that in the $\theta \to 0$ limit the non-commutative theory smoothly tends to the commutative theory. One has to first compute the integral at some θ for arbitrary small external momenta, renormalize the fields and fine-tune the bare parameters in the lagrangian to eliminate any UV cut-off dependence and then study the small non-commutative momentum limit in the resulting expression.

[2]We refer the reader in refs. [4, 3] for the decomposition of diagrams into planar and non-planar ones in double line notation. This is a standard method used in perturbation theory which is particularly useful in analyzing non-commutative field theories. As we mention in the introduction, only non-planar diagrams acquire phases involving internal loop momenta and thus the trigonometric identity we used helps decompose the integral into its planar and non-planar contributions.

[3]In euclidean space the integral is of course real, $\int d^4l/(2\pi)^4 \frac{1}{l^4}\, e^{i\tilde{p}\, l} = \alpha\, (\log L - \log|\tilde{p}|), \;\; \alpha > 0$.

[4]Up to numerical factors, it would be the self energy diagram with charged fermions circulating in the loop.

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The planar piece is divergent but independent of θ. Therefore, it has to be regulated by introducing some ultraviolet cut-off Λ. Naive power counting implies that it quadratically diverges with the cut-off. However, the integral can be regulated using Pauli Villars regularization to eliminate the quadratic divergence. This is a consequence of gauge invariance. The logarithmic dependence on the cut-off can then be eliminated by renormalizing the fields (wave-function renormalization). Then, we are left with an expression which is well behaved in the $\Lambda \to \infty$ limit.

For small but finite non-commutative momentum, the non-planar piece is finite and insensitive to the cut-off Λ. Therefore, in the $\Lambda \to \infty$ limit, it can be computed as we have done above. After we renormalize the fields and take the $\Lambda \to \infty$ limit, we study the small non-commutative momentum limit in the resulting expression, adding the two pieces together. The expression is finite and well behaved in the $\Lambda \to \infty$ limit. At one loop order, the wave-function renormalization introduced to eliminate the cut-off dependence of the planar piece will not change the contributions of the non-planar piece. Then from the non-planar piece we get the singular behavior in θ obtained above. We note that the $\Lambda \to \infty$ limit does not commute with the small non-commutative momentum limit at finite Λ [3], but this indicates the peculiar UV/IR mixing in this theory.

The physical interpretation of terms like eq. (3.6) is very interesting. For small non-commutative momentum, the one-loop inverse propagator (for the renormalized fields) is given by

$$\Gamma_{\mu\nu} = i\left[(p_0^2 - p_3^2 - P^2)g_{\mu\nu} + g^2 c\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^4}\right], \tag{3.7}$$

where P represents the projection of the spatial momentum on the $(1,2)$ plane. From this matrix we can read the dispersion relation for the two physical, transversely polarized photons. Suppose P is along the 2-direction so that $\tilde{P}$ is in the 1-direction. Then the photon polarized in the direction perpendicular to $\tilde{P}$ satisfies the same dispersion relation as a photon would in the commutative theory

$$p_0^2 = p_3^2 + P^2\,. \tag{3.8}$$

However, the photon polarized along the 1-direction, parallel to $\tilde{P}$, satisfies a different dispersion relation given by

$$p_0^2 = p_3^2 + P^2 + cg^2\frac{1}{\theta^2 P^2}\,. \tag{3.9}$$

This splitting of the polarization states of the gauge boson is perfectly consistent with gauge invariance. Indeed the vacuum polarization tensor in eq. (3.6) is purely transverse which follows from the identity $p\tilde{p} = 0$. This effect would not be possible without the breaking of Lorentz invariance caused by θ.

We remark at this point that since the contributions from the scalar loops and from the gauge sector are gauge-invariant by themselves, they individually give combinations of the form (3.4). The computations involving gauge bosons (figure 3) and

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scalars (line (2) on figure 4) in the loop are very similar, and without giving explicit details, we add all three sectors together obtaining

$$i\Pi^{\mu\nu}(p) = 8ig^2(N_s + 2 - 2N_f)\,\alpha\,\frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^4}\,. \tag{3.10}$$

This shows that in the supersymmetric theories with an equal number of bosons and fermions, quadratic divergences in the photon self-energy do not appear at one loop.

As we have mentioned before, the coefficient $c = 8(N_s+2-2N_f)$ of the quadratic anomalous term is proportional to the number of bosons minus the number of fermions in the theory. In particular it is zero in any supersymmetric model. This had to be the case since the splitting of the two photon states would be inconsistent with supersymmetry. In particular, the fermions in the theory remain massless with dispersion relation

$$\not{p} = 0\,. \tag{3.11}$$

Note also that in the $\mathcal{N} = 4$ case the SO(6) R-symmetry of the theory insures that the scalars do not split. So it would be impossible to achieve bose-fermi degeneracy with multiplets of $\mathcal{N} = 4$ supersymmetry had c been non zero.

We also note that the coefficient c will vanish in softly broken supersymmetric theories as well. In particular, the effect we find arises from high momenta circulating the loop. Therefore, it is independent of the mass of the loop particle. As long as the number of fermions equals the number of bosons in the theory c will cancel. Simple dimensional analysis shows that the only effect of adding a mass term in the lagrangian is to modify the logarithmic singularities as $\tilde{p} \to 0$.

Finally, we note that c becomes negative if the number of fermions is bigger than the number of bosons. This is also the case in the ϕ^4 theory when we include fermions coupled to the scalar field through Yukawa couplings. In both cases, the theory becomes unstable at low energies.

4. Vertex corrections

Next we analyze 1-loop corrections to the three-photon vertex. In figures 5 and 6, we have drawn all 1PI Feynman diagrams that contribute corrections to the vertex up to 1-loop order in perturbation theory. As before, we shall see that there are unfamiliar long distance effects arising from the region of integration of high loop momenta.

Let us consider in detail the scalar graph involving cubic vertices only (figure 6, 1st graph on line (1)). Applying the Feynman rules, we find

$$\begin{aligned} -8ig^3 \int \frac{d^4l}{(2\pi)^4} \frac{(2l+p_1)^{\mu_1}(2l-p_2)^{\mu_2}(2l+p_1-p_2)^{\mu_3}}{l^2(l-p_2)^2(l+p_1)^2} \sin\left(\frac{\tilde{p}_1 l}{2}\right) \sin\left(\frac{\tilde{p}_2 l}{2}\right) \times \\ \times \sin\left(\frac{\tilde{p}_3(l+p_1)}{2}\right). \end{aligned} \tag{4.1}$$

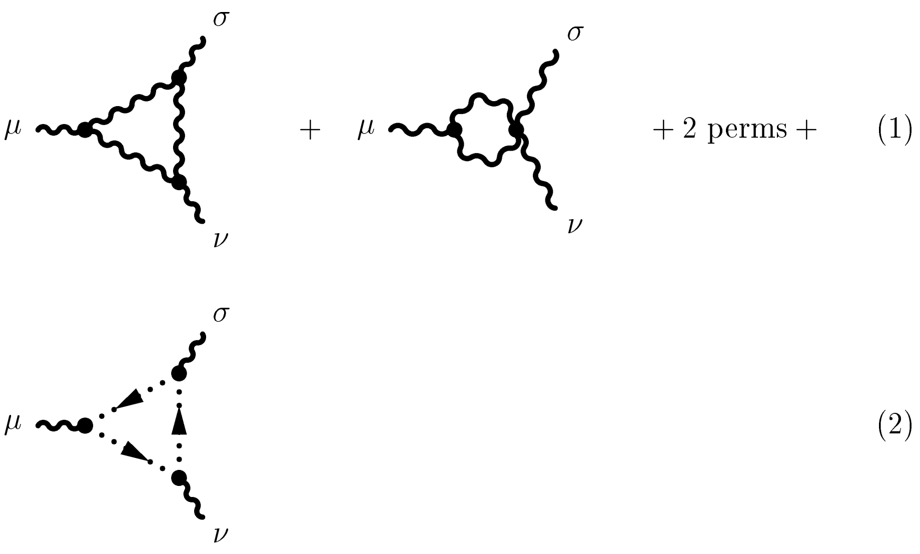

Figure 5: Vertex corrections: gauge sector.

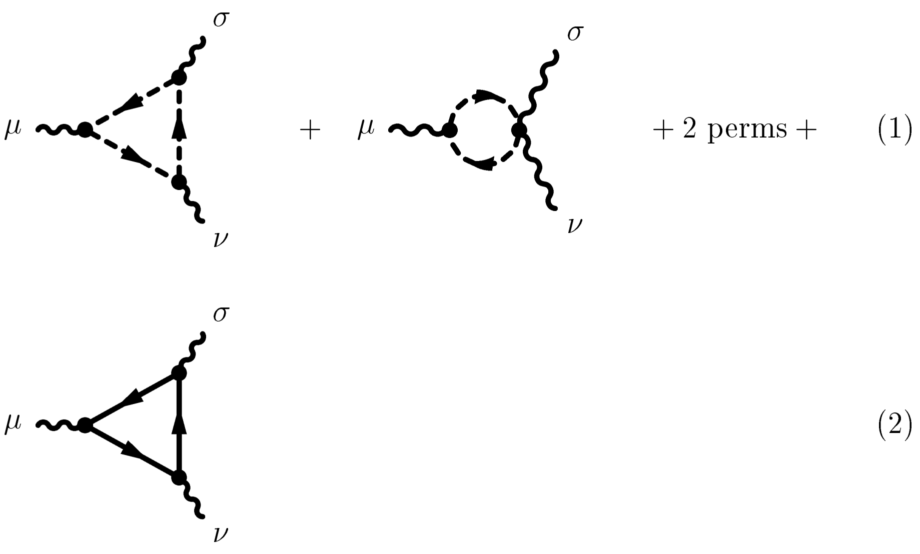

Figure 6: Vertex corrections: matter sector.

We must also impose overall momentum conservation so that

$$p_1 + p_2 + p_3 = 0\,. \tag{4.2}$$

We have amputated the external propagators for compactness. Equation (4.1) has to be multiplied by the number of scalars in the theory. Ignoring the phases for a moment, we note that the integrand is linear in l at high loop momentum

$$\sim \int \frac{d^4l}{(2\pi)^4} \frac{l^{\mu_1} l^{\mu_2} l^{\mu_3}}{l^6}\,. \tag{4.3}$$

In the commutative non-abelian theory, however, no linear divergence arises because the integral is zero by symmetry. Thus it is at most logarithmic in the cutoff. In the non-commutative case the oscillating phases spoil the rotational symmetry. They make the integral finite but they induce low momentum poles of the form

$$\frac{\tilde{p}^{\mu_1}\tilde{p}^{\mu_2}\tilde{p}^{\mu_3}}{\tilde{p}^4}\,. \tag{4.4}$$

Unlike the previous effect which was of order θ^{-2}, this is a θ^{-1} effect.

To compute the precise coefficient in front of such anomalous terms, it is enough to consider high momentum running in the loop ignoring external momenta in the denominators. Then the integral becomes

$$-8ig^3\cos\left(\frac{\tilde{p}_3 p_1}{2}\right)\int\frac{d^4l}{(2\pi)^4}\frac{8l^{\mu_1}l^{\mu_2}l^{\mu_3}}{l^6}\sin\left(\frac{\tilde{p}_1 l}{2}\right)\sin\left(\frac{\tilde{p}_2 l}{2}\right)\sin\left(\frac{\tilde{p}_3 l}{2}\right). \tag{4.5}$$

To carry out this integral, it is useful to express the product of sines as a sum of exponentials. Using eq. (4.2), we can write the product of sines as

$$-\frac{1}{4}\left[\sin(\tilde{p}_1 l)+\sin(\tilde{p}_2 l)+\sin(\tilde{p}_3 l)\right]. \tag{4.6}$$

We are left with a sum of three simpler integrals, and, in addition, we can replace each sine by an exponential in the integral. In this form, it is easy to see that the contributions arise solely from the six non-planar graphs in the double line notation.

To these, we must add the contributions from the scalar graphs that involve a quartic vertex (Second graph and its two permutations on line (1) in figure 6). The graphs produce a different tensor structure but the phase factors are the same. There are three such graphs from permuting the external particles among the external lines. Each graph has to be multiplied by a symmetry factor of 1/2. Adding all four graphs together yields the following integrals:

$$\begin{aligned}4g^3N_s\cos(\frac{\tilde{p}_3 p_1}{2})\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^6}&\left[4l^{\mu_1}l^{\mu_2}l^{\mu_3}-l^2(l^{\mu_1}g^{\mu_2\mu_3}+\text{perms})\right]\times\\ &\times\ (e^{i\tilde{p}_1 l}+e^{i\tilde{p}_2 l}+e^{i\tilde{p}_3 l})\,.\end{aligned} \tag{4.7}$$

The pure gauge sector graphs and the fermionic graphs can be computed in the same way. One has to remember to include a combinatorics factor of 2 for the ghosts and fermion graphs arising from the two different cyclic orderings of the external particles on the loop. The gauge boson quartic graph has a symmetry factor of a 1/2. Including these has the effect of changing the coefficient in front of the integrals to

$$N_s+2-2N_f\,, \tag{4.8}$$

where N_f is the number of Majorana fermions in the theory. We see that in any supersymmetric theory the linear poles are absent.

We will now explicitly find the structure of the linear poles. To compute the integral in (4.7),

$$I^{\mu_1\mu_2\mu_3} = \int \frac{d^4 l}{(2\pi)^4} \frac{(4l^{\mu_1}l^{\mu_2}l^{\mu_3} - l^2(l^{\mu_1}g^{\mu_2\mu_3} + l^{\mu_2}g^{\mu_1\mu_3} + l^{\mu_3}g^{\mu_1\mu_2}))}{l^6}\, e^{i\tilde{p}\,l}\,, \tag{4.9}$$

we again use $\int d^4l/(2\pi)^4 \frac{1}{l^4}\, e^{i\tilde{p}\,l} = i\alpha\,(\log L - \log|\tilde{p}|)$. Notice that (4.9) is convergent, and the L-dependent part can thus be dropped. We can write

$$I^{\mu_1\mu_2\mu_3} = i\,(4\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3} - \Box(g^{\mu_2\mu_3}\partial^{\mu_1} + g^{\mu_1\mu_3}\partial^{\mu_2} + g^{\mu_1\mu_2}\partial^{\mu_3}))\;\; J(\tilde{p})\,, \tag{4.10}$$

where $J(\tilde{p}) = \int d^4l/(2\pi)^4 \frac{1}{l^6}\, e^{i\tilde{p}\,l} = -iJ_E(\tilde{p})$. The integral over euclidean space, $J_E(\tilde{p})$, is obtained from $J(\tilde{p})$ after performing a Wick rotation. It satisfies

$$\Box\, J_E(\tilde{p}) = \frac{1}{\tilde{p}^3}\,\frac{d}{d\tilde{p}}\,\tilde{p}^3\,\frac{d}{d\tilde{p}}\; J_E(\tilde{p}) = \alpha\;\log|\tilde{p}|\,. \tag{4.11}$$

Solving for J and substituting

$$J(\tilde{p}) = -iJ_E(\tilde{p}) = -\frac{i\alpha}{32}\,\tilde{p}^2\,(4\,\log\tilde{p} - 3) \tag{4.12}$$

in (4.10) we find

$$I^{\mu_1\mu_2\mu_3} = -2\alpha\,\frac{\tilde{p}^{\mu_1}\tilde{p}^{\mu_2}\tilde{p}^{\mu_3}}{\tilde{p}^4}\,. \tag{4.13}$$

Note that the second possible tensor structure, $(\tilde{p}^{\mu_1}g^{\mu_2\mu_3} + \tilde{p}^{\mu_2}g^{\mu_1\mu_3} + \tilde{p}^{\mu_3}g^{\mu_1\mu_2})/\tilde{p}^2$, cancelled completely, just like the photon mass term in the photon self-energy correction (3.6). This fact will turn out to be essential in showing the gauge invariance of the S-matrix in the next section.

We can now summarize the computations in this section by giving the linearly divergent terms in the correction to the 3-point photon vertex:

$$\Gamma^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = -8\alpha g^3\,(N_s+2-2N_f)\,\cos\left(\frac{1}{2}\tilde{p}_3 p_1\right)\,\sum_{i=1}^{3}\frac{\tilde{p}_i^{\mu_1}\tilde{p}_i^{\mu_2}\tilde{p}_i^{\mu_3}}{\tilde{p}_i^4} + \cdots\,, \tag{4.14}$$

where the terms denoted by $\cdots$ are at most logarithmic in $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$.

The anomalous effects found in this paper and in [3] are highly non local but in a particular way. The matrix elements in eqs. (3.10) and (4.14) depend only on the components of momentum in the x^1, x^2 plane and are independent of x^3 and x^4. Thus while non local in the non-commutative directions they are completely local in the commutative directions.

In supersymmetric theories the non-local θ^{-2} and θ^{-1} effects are absent, at least at one loop. However there are still logarithmic dependences on P which are proportional to the one-loop coefficient of the β function[5] in the corresponding commutative non-abelian gauge theory. Once again the corresponding effects are non local only in the non-commutative directions. Theories such as $N = 4$ super Yang-Mills theory with vanishing β function seem to be free of the non-analytic dependence on θ.

[5] The β function is the one controlling the running of the t'Hooft coupling in the large-N limit.

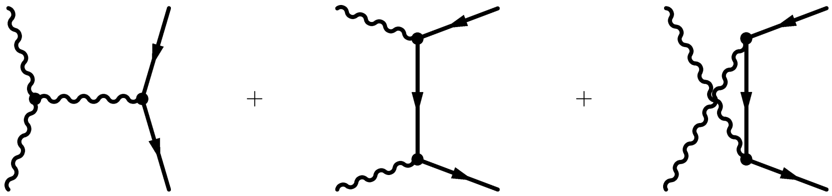

Figure 7: Tree level scattering diagrams.

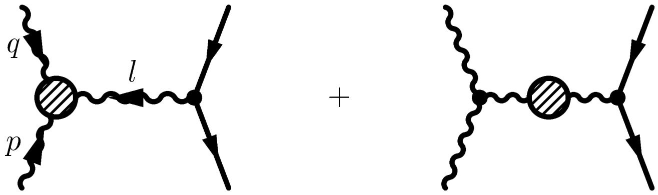

Figure 8: The sum of these diagrams is gauge invariant.

5. Gauge invariant S-matrix?

In this section we show that the anomalous terms in the 2-point and 3-point functions we computed are consistent with gauge invariance. To check gauge invariance, we study a case involving scattering of two gauge bosons into two fermions. We consider 1-loop diagrams only and choose the kinematic variables so that the anomalous terms we found dominate.

All particles must be put on shell using the appropriate, corrected dispersion relations up to 1-loop order in perturbation theory. We choose one of the gauge bosons to be transversely polarized. We denote its polarization by ϵ and its momentum by p; then

$$\epsilon_\mu p^\mu = 0\,. \tag{5.1}$$

In order to test gauge invariance we set the polarization of the other gauge boson, θ, equal to its momentum q. The momenta of the fermions are denoted by (l_1, l_2). Gauge invariance requires this scattering amplitude to be zero. The tree level diagrams contributing to the process are shown in figure 7.

Now we choose a specific kinematic limit. We choose $l_1 + l_2 = -l$ so that $\tilde{l}$ is small. Furthermore, we let l^2 be small so that only the s-channel diagrams are important. Two types of one loop diagrams are important (figure 8), the diagrams involving corrections to the intermediate gauge boson's propagator and the diagrams involving corrections to the three gauge boson vertex. Corrections to the two fermion-gauge boson vertex are at most logarithmic in $\tilde{l}$ and, therefore, sub-leading. We can think of the intermediate gauge boson being coupled to some current j_μ. The exact form of j_μ is not important for our purposes.

First consider 1-loop diagrams involving the gauge boson self-energy. In the limit when $\tilde{l}$ is small the contribution to the amplitude becomes

$$i\mathcal{M}_1 = 16i\alpha g^4 \sin\left(\frac{q\tilde{l}}{2}\right) \epsilon_\nu q_\mu \left[g^{\mu\nu}(q-p)^\rho + g^{\nu\rho}(p-l)^\mu + g^{\rho\mu}(l-q)^\nu\right] \frac{\tilde{l}_\rho(\tilde{l}j)}{l^4\tilde{l}^4}. \quad (5.2)$$

The first factor is a phase factor coming from the 3-photon vertex. Now, using momentum conservation $-l = p+q$, $ep = 0$, $l\tilde{l} = 0$ and $q^2 = 0$, we see that this piece becomes

$$8i\alpha \frac{(\epsilon\tilde{l})(q\tilde{l})(\tilde{l}j)}{l^2\tilde{l}^4}. \quad (5.3)$$

For small $\tilde{l}$, $\sin(q\tilde{l}/2)$ is just $q\tilde{l}/2$. We note that the dispersion relation of a longitudinally polarized photon remains $q^2 = 0$ since the self-energy corrections we found are transverse.

Next we turn to diagrams involving one loop corrections to the vertex. In the limit $\tilde{l} \to 0$, the contribution to the amplitude becomes

$$i\mathcal{M}_2 = -8i\alpha g^4 \frac{(\epsilon\tilde{l})(q\tilde{l})(\tilde{l}j)}{l^2\tilde{l}^4}. \quad (5.4)$$

This is because in this limit the vertex is dominated by anomalous terms of the form $\tilde{l}^\mu\tilde{l}^\nu\tilde{l}^\rho/\tilde{l}^4$. Adding the two contributions together, we see that the amplitude becomes zero as required by gauge invariance.

Next we study the case when $\tilde{p} \to 0$. Again, we study this particular case because we can isolate the possible singular terms. The most dangerous 1-loop diagram contributing to the scattering amplitude is the s-channel diagram involving corrections to the 3-gauge boson vertex. All other diagrams are sub-leading. The 1-loop contribution to the amplitude is then given by

$$i\mathcal{M} = -8i\alpha g^4 \frac{(\epsilon\tilde{p})(q\tilde{p})(\tilde{p}j)}{l^2\tilde{p}^4}. \quad (5.5)$$

We now distinguish between two cases. First we note that if ϵ is perpendicular to $\tilde{p}$, the amplitude is zero. Recall also from the previous section that this is the photon with un-corrected dispersion relation. If on the other hand ϵ is along $\tilde{p}$, the amplitude is not zero but given by

$$i\mathcal{M}_2 = -8i\alpha g^4 \frac{(q\tilde{p})(\tilde{p}j)}{l^2\tilde{p}^2}. \quad (5.6)$$

This contribution is cancelled by the tree level graph once we use the correct dispersion relation for the gauge boson. The tree level graph is given by

$$i\mathcal{M}_3 = -2i\alpha g^2 \sin\left(\frac{q\tilde{p}}{2}\right) \frac{(\epsilon j)(2pq)}{l^2}. \quad (5.7)$$

The first factor is a phase factor. We also used the following relations: $jl = 0$, $\epsilon p = 0$ and $q^2 = 0$. Using momentum conservation, we also find that

$$2pq = l^2 - p^2 \,. \tag{5.8}$$

Now, when the polarization of the gauge boson is along the $\tilde{p}$ direction, the dispersion relation gets corrected as follows:

$$p^2 = 8\alpha g^2 \frac{1}{\tilde{p}^2} \,. \tag{5.9}$$

Therefore, there is order a g^4 contribution from the tree-level graph given by

$$i\mathcal{M}_3 = 8i\alpha g^4 (q\tilde{p}) \frac{(\tilde{p}j)}{l^2 \tilde{p}^2} \,. \tag{5.10}$$

Adding the two together we see that to order g^4 the amplitude is zero as it is required by gauge invariance.

The case involving scattering of scalars is more subtle because the two scalar-gauge boson vertex also contains linear divergences in θ. We defer this case for an upcoming paper [9].

6. Conclusions

The most naive expectation about the non-commutative field theories is that they become commutative when the non-commutativity parameter $\theta^{ij} \to 0$ limit is taken. In other words, star-products are replaced by ordinary products of the fields in the lagrangian in this limit, and one may expect that the theory becomes commutative also at the quantum level. As it was shown in [3] this is generally not true due to the appearance of the new divergences at low non-commutative momenta. In the commutative theories, these divergences appear at high momenta in the superficially divergent loop integrals, but they can be eliminated by an appropriate choice of the regularization scheme. In the non-commutative theories, some of these divergences simply do not occur due to oscillating phases associated with star-products in the vertices which make the integrals finite. The non-commutative momenta thus play the role of the regulator. The dependence on these non-commutative momenta, however, does not disappear and manifests itself in the form of the new infrared divergences at small values of non-commutative momenta. These effects can also be characterized as non-analytic behavior in θ.

A less naive expectation would be that a non-commutative gauge theory must be free of quadratic and linear poles at low non-commutative momenta since the corresponding commutative non-abelian gauge theory contains at most logarithmic divergences. In this paper we have shown that even this expectation does not hold,

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and both quadratic and linear poles appear in a generic gauge theory. The structure of these new poles is consistent with gauge invariance but not Lorentz invariance. These effects are local in the direction perpendicular to the non-commutativity plane and completely non local in the non-commutative directions.

In supersymmetric gauge theories these poles cancel between the bosons and the fermions at the one loop level but even these theories typically contain logarithmic divergences at small values of non-commutative momenta. We expect that in $\mathcal{N} = 4$ SYM theory even the logarithmic divergences do not occur, and thus this theory is completely free of anomalous effects in the small non-commutative momentum limit. This is the only theory that we know that reduces to its commutative counterpart in the limit $\theta^{ij} \to 0$.

Acknowledgments

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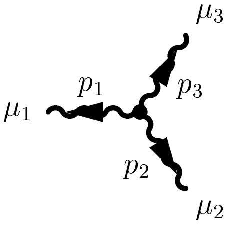
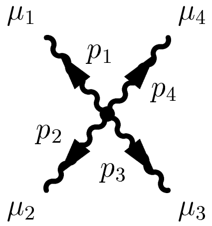
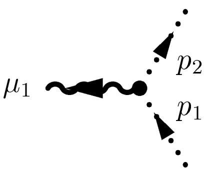
A. Feynman rules



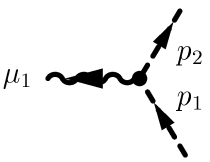
$$= \begin{array}{l} -2g\sin(\tfrac{1}{2}\tilde{p}_1 p_2)\times \\ \left[(p_1-p_2)^{\mu_3} g^{\mu_1\mu_2} + (p_2-p_3)^{\mu_1} g^{\mu_2\mu_3} + (p_3-p_1)^{\mu_2} g^{\mu_3\mu_1}\right] \end{array}$$



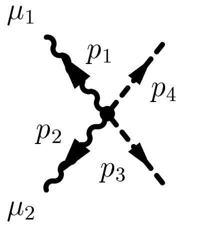
$$= -4ig^2\Big[\begin{array}{l} \sin(\tfrac{1}{2}\tilde{p}_1 p_2)\sin(\tfrac{1}{2}\tilde{p}_3 p_4)(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3}) + \\ \sin(\tfrac{1}{2}\tilde{p}_3 p_1)\sin(\tfrac{1}{2}\tilde{p}_2 p_4)(g^{\mu_1\mu_4}g^{\mu_2\mu_3} - g^{\mu_1\mu_2}g^{\mu_3\mu_4}) + \\ \sin(\tfrac{1}{2}\tilde{p}_1 p_4 \sin(\tfrac{1}{2}\tilde{p}_2 p_3 (g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4})\Big] \end{array}$$



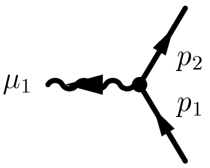
$$= 2gp_2^\mu \sin(\frac{1}{2}\tilde{p}_1 p_2)$$



$$= 2g(p_1^\mu + p_2^\mu)\sin(\frac{1}{2}\tilde{p}_1 p_2)$$



$$= \begin{array}{l} -4ig^2 g^{\mu_1\mu_2} \\ \times\Big[\sin(\tfrac{1}{2}\tilde{p}_1 p_3)\sin(\tfrac{1}{2}\tilde{p}_2 p_4) + \sin(\tfrac{1}{2}\tilde{p}_1 p_4)\sin(\tfrac{1}{2}\tilde{p}_2 p_3)\Big] \end{array}$$



$$= 2g\gamma^\mu \sin(\frac{1}{2}\tilde{p}_1 p_2)$$

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